Adjoint pseudospectral least-squares methods for an elliptic boundary value problem

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Abstract

The adjoint approach for Legendre pseudospectral least-squares methods is presented by adopting the adjoint first-order systems developed in [Z. Cai, T. Manteuffel, S. McCormick, J. Ruge, First-order system LL* (FOSLL*): scalar elliptic partial differential equations, SIAM J. Numer. Anal. 39 (2001) 1418–1445]. The discrete adjoint least-squares functional on a polynomial space using Legendre–Gauss–Lobatto points and weights is shown to be equivalent to $H^1$ norm. The spectral convergence is also provided with several numerical results.

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1. Introduction

The spectral or pseudospectral methods have been known as one of the very accurate and popular methods among other numerical methods for solving partial differential equations (see [1,6,8,9] and [14] for example) such as

\[
\begin{aligned}
- \nabla \cdot A \nabla p + b \cdot \nabla p + cp &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega = [-1, 1] \times [-1, 1]$, $\partial \Omega$ denotes the boundary of $\Omega$, $A$ is a $2 \times 2$ symmetric matrix of bounded functions, $b$ is a $2 \times 1$ vector of functions in $L^2(\Omega)$ such that $\nabla \cdot b \in L^\infty(\Omega)$, $c$ is a given nonnegative bounded function and $f$ is a given continuous function.

It is known that, by introducing a physical meaningful variable such as velocity flux, (1.1) can be written as an equivalent system of first-order partial differential equations, so that it can be minimized in the sense of a least-squares functional. In this respect, the least-squares finite element techniques and theories may be found in, for example, [2–4] and [10] for approximate solution of the given system of partial differential equations. On the other hand, the least-
squares approach using pseudospectral methods was presented in [13] for (1.1) with constant matrix, vector and scalar coefficients and Stokes problem in [12] where the $L^2$ norm of least-squares functionals were used to yield $H^1$ equivalence. This $L^2$ norm least-squares approach was restricted to an elliptic boundary value problem with constant advection–diffusion–reaction coefficient terms which leads not only to use of its sufficient regularity of a solution but also to use of equivalence of numerical quadrature easily. But, with lack of enough regularity of a solution for a given boundary value problem, the spectral collocation least-squares methods may require some other considerations for an existence and uniqueness of an approximate solution which may not be asked to compute derivatives of the advection–diffusion–reaction coefficients. Fortunately, this technique has been developed as the adjoint approach, so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach has one of the merits which possesses the full efficiency of the so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach also has the trivial solution.

The outline of this paper can be described as follows. In Section 2, some preliminaries, definitions, notations and spaces are presented. In Section 3, we review the ideas and concepts of FOSLL* methodology developed in [5] and [11] for general elliptic boundary value problem with a mixed boundary condition. In Section 4, with the adjoint problem also has the trivial solution.

The adjoint least-squares finite element approach has been applied to the Legendre pseudospectral methods for a second-order elliptic boundary value problem (1.1). We assume that $\lambda_{max} \cdot \lambda_{min} \Rightarrow 0$ or $\lambda_{max} \cdot \lambda_{min} \Rightarrow 1$ and almost all $(x, y) \in \Omega$. We assume that (1.1) has the trivial solution when $f = 0$ and its homogeneous adjoint problem also has the trivial solution.

2. Preliminaries, definitions, notations and spaces

The standard notations and definitions are used for the Sobolev spaces $H^1_0(\Omega)$ and $H^1(\Omega)^2$ equipped with inner products $(\cdot, \cdot)$, and corresponding norms $\| \cdot \|_s$, $s \geq 0$. The space $H^0(\Omega)$ coincides with $L^2(\Omega)$, in which case the norm and inner product will be denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, respectively. For a two dimensional vector $v = (v_1, v_2)$, the divergence and curl will be denotes as $\nabla \cdot v = \partial_x v_1 + \partial_y v_2$ and $\nabla \times v = -\partial_y v_1 + \partial_x v_2$ respectively. We also use the notations $H(\nabla \cdot; \Omega)$ and $H(\nabla \times; \Omega)$ with the norms $\|v\|_{H(\nabla \cdot; \Omega)} := (\|v\|^2 + \|\nabla \cdot v\|^2)^{1/2}$ and $\|v\|_{H(\nabla \times; \Omega)} := (\|v\|^2 + \|\nabla \times v\|^2)^{1/2}$ respectively. The space $H_0(\nabla \times; \Omega)$ denotes the set of all functions $w$ in $H(\nabla \times; \Omega)$ with the boundary condition $\mathbf{t} \cdot \mathbf{w} = 0$ on $\Gamma$. Define other spaces as

$$H^1_B(\Omega) := \{u \in H^1(\Omega); \quad u = 0 \text{ on } \Gamma_B\},$$

$$H_B(\nabla \cdot; \Omega) := \{u \in H^2(\Omega)^2; \quad \nabla \cdot (Cu) \in H^1(\Omega), \quad \mathbf{n} \cdot Cu = 0 \text{ on } \Gamma_B\},$$

$$H_B(\nabla \times; \Omega) := \{u \in H^2(\Omega)^2; \quad \nabla \times (Cu) \in H^1(\Omega), \quad \mathbf{t} \cdot Cu = 0 \text{ on } \Gamma_B\},$$

where $C$ is a matrix such as $A$, $A^1$ or $A^{-1}$ and $B$ is either Dirichlet boundary $D$ or Neumann boundary $N$. Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to $N$. Let $\{\xi_i\}_{i=0}^N$ be the LGL points on $[-1, 1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1$, which are the zeros of $(1 - t^2)\xi_N(t)$ where $\xi_N$ is the $N$th Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by $w_j = 2/(N(N+1)\xi_N(\xi_j)^2)$ for $1 \leq j \leq N - 1$ and $w_0 = w_N = 2/(N(N+1)\xi_N(\xi_0)^2)$. Then, we have the following LGL quadrature formula such that $\int_{-1}^1 p(t) dt = \sum_{i=0}^N w_i p(\xi_i)$, $\forall p \in \mathcal{P}_{2N-1}$. Let $\{\phi_i\}_{i=0}^N$ be the set of Lagrange polynomials of degree $N$ with respect to LGL points $\{\xi_i\}_{i=0}^N$ which satisfy $\phi_i(\xi_j) = \delta_{ij}$, $\forall i, j = 0, 1, \ldots, N$. The 2 norm of least-squares functionals were used to yield $H^1$ equivalence. This $L^2$ norm least-squares approach was restricted to an elliptic boundary value problem with constant advection–diffusion–reaction coefficient terms which leads not only to use of its sufficient regularity of a solution but also to use of equivalence of numerical quadrature easily. But, with lack of enough regularity of a solution for a given boundary value problem, the spectral collocation least-squares methods may require some other considerations for an existence and uniqueness of an approximate solution which may not be asked to compute derivatives of the advection–diffusion–reaction coefficients. Fortunately, this technique has been developed as the adjoint approach, so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach has one of the merits which possesses the full efficiency of the so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach also has the trivial solution.

The outline of this paper can be described as follows. In Section 2, some preliminaries, definitions, notations and spaces are presented. In Section 3, we review the ideas and concepts of FOSLL* methodology developed in [5] and [11] for general elliptic boundary value problem with a mixed boundary condition. In Section 4, with the adjoint problem also has the trivial solution. Fortunately, this technique has been developed as the adjoint approach, so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach has one of the merits which possesses the full efficiency of the so called, FOSLL* found in [5] and [11]. It has been known that the adjoint least-squares finite element approach also has the trivial solution.

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where $\delta_{ij}$ denotes the Kronecker delta. The two-dimensional LGL nodes $\{x_{ij}\}$ and weights $\{w_{ij}\}$ are given by
\[
x_{ij} = (\xi_i, \xi_j), \quad w_{ij} = w_i w_j, \quad i, j = 0, 1, \ldots, N.
\]
Let $Q_N$ be the space of all polynomials of degree less than or equal to $N$ with respect to each single variable $x$ and $y$. Define the basis for $Q_N$ as
\[
\psi_{ij}(x, y) = \phi_i(x) \phi_j(y), \quad i, j = 0, 1, \ldots, N.
\]
For any functions $u$ and $v$ defined on $\bar{\Omega}$, the discrete scalar product and associated norm are given by
\[
\langle u, v \rangle_N = \sum_{i,j=0}^{N} w_{ij} u(x_{ij}) v(x_{ij}) \quad \text{and} \quad \|v\|_N = \langle v, v \rangle_N^{1/2}.
\]
Then, the exactness of one dimensional numerical quadrature yields that
\[
\langle u, v \rangle_N = (u, v) \quad \text{for} \quad uv \in Q_{2^{2N}}.
\]
It is well known that
\[
\|v\| \leq \|v\|_N \leq \gamma^* \|v\|, \quad \forall v \in Q_N,
\]
where $\gamma^* = 2 + \frac{1}{N}$. For any continuous function $v$ on $\bar{\Omega}$, we denote by $I_N v \in Q_N$ the interpolant of $v$ at the LGL-points $\{x_{ij}\}$ such that
\[
I_N v(x) = \sum_{i,j=0}^{N} v(x_{ij}) \psi_{ij}(x), \quad \forall x \in \bar{\Omega}.
\]
The interpolation error estimate is known (see [1,6,14]) as
\[
\|v - I_N v\|_k \leq C N^{-s} \|v\|_s, \quad k = 0, 1,
\]
provided $v \in H^s(\Omega)$ for $s \geq 2$ and it is also known that for any $u \in H^s(\Omega)$, $s \geq 2$, and any $v_N \in Q_N$
\[
|(u, v_N) - \langle u, v_N \rangle_N| \leq C N^{-s} \|u\|_s \|v_N\|.
\]
For $u \in H^s(\Omega)$, and let $P_N u$ be the $L^2(\Omega)$ orthogonal projection of $u$ onto $Q_N$. Then
\[
\|u - P_N u\|_k \leq C N^{\epsilon(k,s)} \|u\|_s,
\]
where $\epsilon(k,s) = 2k - s - \frac{1}{2}$ if $1 \leq k \leq s$, or $\epsilon(k,s) = \frac{3k}{2} - s$ if $0 \leq k \leq 1$. The approximation result (2.7) can be found in [7].

**Remark 1.** Even if the theoretical analysis is not done in this paper for a general elliptic boundary value problem (1.1) with a mixed boundary condition
\[
p = 0 \quad \text{on} \quad \Gamma_D, \quad \text{and} \quad \mathbf{n} \cdot \nabla p = 0 \quad \text{on} \quad \Gamma_N
\]
where $\partial \Omega = \Gamma_D \cup \Gamma_N$ and $\mathbf{n}$ is the outward unit vector normal to the boundary, the *adjoint* least-squares numerical demonstrations will be presented in Section 5 for this case. For both theoretical understanding and numerical demonstrations, the adjoint approach for a general elliptic boundary value problem will be reviewed in the following section.
3. First-order systems, continuous least-squares functionals

In this section, we review the known adjoint approach for solving the given boundary value problem (1.1) for a general understanding. Let \( u = A^{\frac{1}{2}} \nabla p \). Then Eq. (1.1) becomes

\[
\begin{align*}
L(u, p) & := \begin{cases} 
A^{-\frac{1}{2}} u - \nabla p = 0 & \text{in } \Omega, \\
-\nabla \cdot A^{\frac{1}{2}} u + b \cdot A^{-\frac{1}{2}} u + cp = f & \text{in } \Omega, \\
\nabla \times A^{-\frac{1}{2}} u = 0 & \text{in } \Omega,
\end{cases}
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
\mathbf{n} \cdot A^{\frac{1}{2}} u = 0 & \quad \text{on } \Gamma_N, \\
\mathbf{n} \times A^{-\frac{1}{2}} u = 0 & \quad \text{on } \Gamma_D, \\
p = 0 & \quad \text{on } \Gamma_D.
\end{align*}
\]

In order to apply the adjoint approach for solving \( L(u, p) = (0, f, 0)^t \) in the sense of least-squares, we need to construct an expanded operator \( L_e \) of \( L \) so that

- both \( L_e \) and its \( L_2 \) adjoint \( L^*_e \) are bijective,
- both \( L_e^{-1} \) and \( (L^*_e)^{-1} \) are bijective.

Then the weak problem to find an \( x \in \mathcal{D}^*_e \) satisfying

\[
(L^*_e x, L^*_e y)_V_1 = (f, y)_V_2 \quad \text{for all } y \in \mathcal{D}^*_e
\]

yields that \( w = L^*_e x \) is the unique solution of \( L_e w = f \), where \( \mathcal{D}^*_e \) is the domain of \( L^*_e \).

Due to the above strategy, by adding a slack variable \( q \), (3.1) becomes a square system:

\[
L_e(u, p, q)^t = (0, -f, 0)^t \quad \text{in } \Omega,
\]

where

\[
L_e := \begin{bmatrix}
A^{-\frac{1}{2}} & -\nabla & -\nabla^\perp \\
\nabla \cdot A^{\frac{1}{2}} - b \cdot A^{-\frac{1}{2}} & -c & 0 \\
-\nabla \times A^{-\frac{1}{2}} & 0 & 0
\end{bmatrix},
\]

with boundary conditions

\[
\begin{align*}
\tau \cdot A^{-\frac{1}{2}} u = 0 & \quad \text{on } \Gamma_D, \\
\mathbf{n} \cdot A^{\frac{1}{2}} u = 0 & \quad \text{on } \Gamma_N, \\
p = 0 & \quad \text{on } \Gamma_D, \\
q = 0 & \quad \text{on } \Gamma_N.
\end{align*}
\]

The domain \( \mathcal{D}_e \) of \( L_e \) is

\[
\mathcal{D}_e := (H_N(\nabla \cdot A^{\frac{1}{2}}; \Omega) \cap H_D(\nabla \times A^{-\frac{1}{2}}; \Omega)) \times H_D^1(\Omega) \times H_N^1(\Omega),
\]

which is a Hilbert space under the norm

\[
\|(u, p, q)\|_{L_e}^2 := \|u\|^2 + \|\nabla \cdot (A^{\frac{1}{2}} u)\|^2 + \|\nabla \times (A^{-\frac{1}{2}} u)\|^2 + \|p\|^2 + \|q\|^2.
\]

The FOSLL* approach is to approximate the solution \( (w, r, s) \) of the corresponding dual problem

\[
L^*_e(w, r, s) = (A^{\frac{1}{2}} \nabla p, p, 0)^t = (u, p, q)^t \quad \text{in } \Omega,
\]

with
where boundary conditions

\[
\begin{align*}
\tau \cdot w &= 0 \quad \text{on } \Gamma_D, \\
n \cdot w &= 0 \quad \text{on } \Gamma_N, \\
r &= 0 \quad \text{on } \Gamma_D, \\
s &= 0 \quad \text{on } \Gamma_N.
\end{align*}
\]

The domain \( \mathcal{D}_e \) of \( \mathcal{L}_e \) is

\[
\mathcal{D}_e := (H_N(\nabla; \Omega) \cap H_D(\nabla \times; \Omega)) \times H^1_D(\Omega) \times H^1_N(\Omega),
\]

which is a Hilbert space under the norm

\[
\| (w, r, s) \|^2 := (\|w\|^2 + \|\nabla \cdot w\|^2 + \|\nabla \times w\|^2)^{\frac{1}{2}} + \|r\|^2 + \|s\|^2.
\]

Now we can recall the well-posedness for the operator \( \mathcal{L} \) and \( \mathcal{L}_e^* \) in the following theorem in [11].

**Theorem 3.1.** Operators \( \mathcal{L}_e \) and \( \mathcal{L}_e^* \) are bijective from \( \mathcal{D} \) and \( \mathcal{D}_e^* \), respectively onto \( (L^2(\Omega))^4 \). Further, \( \mathcal{L}_e \) and \( \mathcal{L}_e^* \) are coercive and continuous in the norms defined in (3.6) and (3.11), respectively.

**Proof.** See Theorem 2.2 in [11]. \(\square\)

Because of the above theorem, for \((u, p, q) \in (L^2(\Omega))^4\) there exists an element \((w, r, s) \in \mathcal{D}_e^*\) such that \((u, p, q)^t = \mathcal{L}_e^*(w, r, s)\). Then we can define the least-squares functional for the system (3.7) as

\[
G_e^*(v, \phi, \psi) = \| \mathcal{L}_e^*(v, \phi, \psi) - (u, p, q)^t \|^2
\]

for \((v, \phi, \psi) \in \mathcal{D}_e^*\). The first-order system least-squares variational problem for (3.7) is to minimize the quadratic functional \(G_e^*(v, \phi, \psi)\) over \(\mathcal{D}_e^*\): find \((w, r, s)^t\) such that

\[
(w, r, s)^t = \arg\min_{(v, \phi, \psi) \in \mathcal{D}_e^*} G_e^*(v, \phi, \psi).
\]

The corresponding variational problem is to find \((w, r, s)^t \in \mathcal{D}_e^*\) such that

\[
a_e^*(((w, r, s); (\hat{w}, \hat{r}, \hat{s})) = \left( (u, p, q)^t, \mathcal{L}_e^*(\hat{w}, \hat{r}, \hat{s}) \right) \quad \forall (\hat{w}, \hat{r}, \hat{s}) \in \mathcal{D}_e^*,
\]

where the bilinear form \(a_e^*(\cdot; \cdot)\) is given by

\[
a_e^*((w, r, s); (\hat{w}, \hat{r}, \hat{s})) = \left( \mathcal{L}_e^*(w, r, s), \mathcal{L}_e^*(\hat{w}, \hat{r}, \hat{s}) \right).
\]

Because of Theorem 3.1, the weak formulation (3.14) yields

\[
(u, p, q)^t = \mathcal{L}_e^*(w, r, s),
\]

which is the unique solution of \( \mathcal{L}_e(u, p, q) = (0, -f, 0)^t \). Hence the right-hand side of (3.14) becomes

\[
((u, p, q)^t, \mathcal{L}_e^*(\hat{w}, \hat{r}, \hat{s})) = ((0, -f, 0)^t, (\hat{w}, \hat{r}, \hat{s})).
\]

In particular, for the problem (1.1) one can modify the operators \( \mathcal{L}_e \) and \( \mathcal{L}_e^* \) with corresponding boundary conditions (3.4) and (3.9), domains (3.10) and norms (3.11) easily. Then the domain and boundary conditions for \( \mathcal{L}_e^* \) will be

\[
\mathcal{D}_e^* := (H(\nabla; \Omega) \cap H_0(\nabla \times; \Omega)) \times H^1_D(\Omega) \times H^1_N(\Omega),
\]

and

\[
\begin{align*}
\tau \cdot w &= 0 \quad \text{on } \Gamma, \\
r &= 0 \quad \text{on } \Gamma_D, \\
s &= 0 \quad \text{on } \Gamma_N.
\end{align*}
\]
Even if the general proof for the ellipticity and continuity of the functional $G^*(w, r, s; 0)$ can be found in [5] and [11] (see Theorem 3.1), the coercivity of $L^e_0$ for the case (1.1) with $b = 0$ and $c = 0$ can be shown in an elementary way below. Note that the corresponding homogeneous FOSLL* least-squares functional defined on $\mathcal{D}^e_0$ in (3.17) becomes

$$G^*(w, r, s; 0) = \|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r - A^{-\frac{1}{2}}\nabla \perp s\|^2 + \|\nabla \cdot w\|^2 + \|\nabla \times w\|^2.$$  

(3.19)

**Theorem 3.2.** Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$. Then, there is a positive constant $C$ such that

$$C(\|w\|_1^2 + \|r\|_1^2 + \|s\|_1^2) \leq G^*(w, r, s; 0).$$  

(3.20)

**Proof.** First note that for any $w \in H(\nabla \cdot; \Omega) \cap H_0(\nabla \times; \Omega)$,

$$\|w\|_1 \leq C(\|\nabla \cdot w\| + \|\nabla \times w\|),$$  

(3.21)

which implies

$$\|w\|_1 \leq CG^*(w, r, s; 0).$$  

(3.22)

Next, we will show

$$\|w\|_1^2 + \|r\|_1^2 \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + G^*(w, r, s; 0)),$$  

(3.23)

where $C$ is an absolute positive constant. Using Poincaré inequality, the uniform ellipticity (1.2) of $A$, triangle inequality and (3.22), it follows that

$$\|r\|_1^2 \leq C\|\nabla r\|^2 \leq C\|A^{\frac{1}{2}}\nabla r\|^2 \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + \|A^{-\frac{1}{2}}w\|^2) \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + \|A^{-\frac{1}{2}}w\|^2 + \|s\|_1^2) \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + G^*(w, r, s; 0)).$$

This completes (3.23). Using the triangle inequality and (3.23), it follows that

$$\|w\|_1^2 + \|r\|_1^2 \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + G^*(w, r, s; 0)) \leq C(\|A^{-\frac{1}{2}}w - A^{-\frac{1}{2}}\nabla r\|^2 + G^*(w, r, s; 0)) + \|A^{-\frac{1}{2}}\nabla \perp s\|^2,$$

(3.24)

where $C$ is an absolute constant. Now it is enough to show

$$\|A^{-\frac{1}{2}}\nabla \perp s\|^2 \leq CG^*(w, r, s; 0)$$  

(3.25)

because the uniform ellipticity (1.2) of $A$, and Poincare inequality leads to

$$\|s\|_1^2 \leq CG^*(w, r, s; 0).$$  

(3.26)

The validity of (3.25) follows in the following fashion.

$$\|A^{-\frac{1}{2}}\nabla \perp s\|^2 = (A^{-\frac{1}{2}}\nabla \perp s, A^{-\frac{1}{2}}\nabla \perp s) = (A^{-\frac{1}{2}}\nabla \perp s, A^{-\frac{1}{2}}(A^{-\frac{1}{2}} \nabla r - A^{-\frac{1}{2}}\nabla \perp s) - (A^{-\frac{1}{2}}\nabla \perp s, w)) \leq C\|A^{-\frac{1}{2}}\nabla \perp s\|^2 G^*(w, r, s; 0)^\frac{1}{2},$$

which leads to (3.26). Now combining all estimates of (3.24) and (3.26) completes the proof of lower bound in (3.20).
4. Adjoint Legendre pseudospectral least-squares methods

In this section, we investigate the adjoint least-squares methods combined with Legendre pseudospectral methods for the first-order system of linear equations equivalent to the problem (3.14) corresponding to (1.1). Let us define the following polynomial spaces as

\[ W_N := [Q_N]^2 \cap (H(\nabla \cdot; \Omega) \times H_0(\nabla \times; \Omega)), \quad R_N = S_N := Q_N \cap H_0^1(\Omega). \]

The discrete least-squares problem associated to (3.14) is to find \((\hat{w}_N, \hat{r}_N, \hat{s}_N) \in W_N \times R_N \times S_N\) such that for all \((\hat{w}_N, \hat{r}_N, \hat{s}_N) \in W_N \times R_N \times S_N,\)

\[ a_e^*(\hat{w}_N, r_N, s_N) = (0, -f, 0) = (\hat{w}_N, \hat{r}_N, \hat{s}_N) \in W_N \times R_N \times S_N, \]

where the discrete bilinear form \(a_e^*(\cdot; \cdot)\) is defined as

\[ a_e^*(\hat{w}_N, r_N, s_N) = (\hat{w}_N, \hat{r}_N, \hat{s}_N) \in W_N \times R_N \times S_N, \]

The corresponding quadratic least-squares functional \(G_e^*(\hat{w}_N, r_N, s_N)\) is then given by

\[ G_e^*(\hat{w}_N, r_N, s_N) = \| \hat{w}_N, r_N, s_N \|_N^2. \]

Lemma 4.1. For any \(u_N \in [Q_N]^2\), it follows that

\[ \lambda \langle u_N, u_N \rangle_N \leq \langle A u_N, u_N \rangle_N \leq \Lambda \langle u_N, u_N \rangle_N, \]

where \(\lambda\) and \(\Lambda\) are the constants in (1.2).

Proof. Since

\[ \langle A u_N, u_N \rangle_N = \sum_{i,j=0}^N u_N^t(x_{ij})A(x_{ij})u_N(x_{ij})w_{ij}, \]

using the inequality (1.2) yields

\[ \lambda \sum_{i,j=0}^N u_N^t(x_{ij})u_N(x_{ij})w_{ij} \leq \langle A u_N, u_N \rangle_N \leq \Lambda \sum_{i,j=0}^N u_N^t(x_{ij})u_N(x_{ij})w_{ij}, \]

which completes the conclusion.

Consider the following bilinear form of the variational problem (3.14)

\[ a_e^*((w, r, s); (\hat{w}, \hat{r}, \hat{s})) = (\hat{w}, \hat{r}, \hat{s}). \]

Then there is a positive constant \(C\) (see [5]) such that

\[ \frac{1}{C} \| (w, r, s) \|_{L^2_e}^2 \leq a_e^*((w, r, s); (w, r, s)) \leq \| (w, r, s) \|_{L^2_e}^2. \]

Lemma 4.2. Let \(\Omega := [-1, 1] \times [-1, 1]\). Then we have for any \(w_N \in (H(\nabla \cdot; \Omega) \cap H_0(\nabla \times; \Omega)) \cap [Q_N]^2,\)

\[ \| w_N \|_1 \leq C(\| \nabla \cdot w_N \|_N + \| \nabla \times w_N \|_N), \]

where \(C\) is a positive constant.

Proof. Combining (3.21) and the equivalence of numerical quadrature leads to the conclusion.
Lemma 4.3. It follows that $\langle \nabla s_N, \nabla r_N \rangle_N = 0$ for any $s_N, r_N \in Q_N \cap H^1_0(\Omega)$.

Proof. The exactness of numerical quadrature and Green’s formula leads to $\langle \nabla s_N, \nabla r_N \rangle_N = (\nabla s_N, \nabla r_N) = 0$. □

In the following proposition, we will show the equivalence over $W_N \times \mathcal{R}_N \times \mathcal{S}_N$ between the norm

$$\| (w_N, r_N, s_N) \|_{L^2}^2$$

and the homogeneous least-squares functional

$$G^*_N(w_N, r_N, s_N; 0) = \| A^{-\frac{1}{2}}w_N - A^\frac{1}{2}r_N - A^{-\frac{1}{2}}\nabla s_N \|_N^2 + \| \nabla \cdot w_N \|_N^2 + \| \nabla \times w_N \|_N^2.$$

For the rest of this section, we assume that $b = 0$ and $c = 0$ in (1.1). Then the existence and convergence of polynomial solution to the adjoint pseudospectral least-squares problem (4.1) can be provided.

Theorem 4.4. For any $(w_N, r_N, s_N) \in W_N \times \mathcal{R}_N \times \mathcal{S}_N$, there exists an absolute constant $C$ such that

$$\frac{1}{C} \| (w_N, r_N, s_N) \|_{L^2} \leq G^*_N(w_N, r_N, s_N; 0) \leq C \| (w_N, r_N, s_N) \|_{L^2}. \quad (4.8)$$

Here, note that

$$\| (w_N, r_N, s_N) \|_{L^2}^2 := \| w_N \|_1^2 + \| r_N \|_1^2 + \| s_N \|_1^2. \quad (4.9)$$

Proof. First one may easily verify the validity of the upper bound in (4.8) by using the triangle inequality, Lemma 4.1 and the equivalence of numerical quadrature. For a lower bound of (4.8), we note that from (4.7)

$$\| w_N \|_1 \leq CG^*_N(w_N, r_N, s_N; 0). \quad (4.10)$$

Next, we will show

$$\| w_N \|_1^2 + \| r_N \|_1^2 \leq C \left( \| A^{-\frac{1}{2}}w_N - A^\frac{1}{2}r_N \|_N^2 + G^*_N(w_N, r_N, s_N; 0) \right). \quad (4.11)$$

Using Poincare inequality, the equivalence of numerical quadrature, triangle inequality, Lemma 4.1, (2) in Lemma 4.3 and (4.7), it follows that

$$\| r_N \|_1^2 \leq C \| \nabla r_N \|_1^2 \leq C \left( \| A^{-\frac{1}{2}}w_N - A^\frac{1}{2}r_N \|_N^2 + \| A^{-\frac{1}{2}}w_N \|_N^2 \right) \leq C \left( \| A^{-\frac{1}{2}}w_N - A^\frac{1}{2}r_N \|_N^2 + \| w_N \|_N^2 \right) \leq C \left( \| A^{-\frac{1}{2}}w_N - A^\frac{1}{2}r_N \|_N^2 + G^*_N(w_N, r_N, s_N; 0) \right),$$

where we used (4.10) in the last inequality. This argument with (4.10) completes (4.11). Now using the triangle inequality and (4.11), it follows that

$$\| w_N \|_1^2 + \| r_N \|_1^2 \leq C \left( G^*_N(w_N, r_N, s_N; 0) + \| A^{-\frac{1}{2}}\nabla s_N \|_N^2 \right). \quad (4.12)$$

Hence, it is enough to show

$$\| A^{-\frac{1}{2}}\nabla s_N \|_N^2 \leq C G^*_N(w_N, r_N, s_N; 0) \quad (4.13)$$

because Lemma 4.1, the equivalence of numerical quadrature and Poincare inequality leads to

$$\| s_N \|_1^2 \leq C G^*_N(w_N, r_N, s_N; 0). \quad (4.14)$$

In fact, due to Lemma 4.3, the equivalence of numerical quadrature and Lemma 4.2, we have
The given first-order system of differential equation (3.2) is given by (see (3.14)).

Hence we may consider the approximate solution (3.14) for some \( A \) and (3.13).

\[
\|A^{-\frac{1}{2}} \nabla \cdot s_N\|_N^2 = (A^{-\frac{1}{2}} \nabla \cdot s_N, A^{-\frac{1}{2}} \nabla \cdot s_N)_N \\
= (A^{-\frac{1}{2}} \nabla \cdot s_N, A^{-\frac{1}{2}} w - A^{\frac{1}{2}} \nabla r_N - A^{-\frac{1}{2}} \nabla \cdot s_N)_N - (A^{-1} \nabla \cdot s_N, w_N)_N \\
\leq C \|A^{-\frac{1}{2}} \nabla \cdot s_N\|_N \|G'_N(w_N, r_N, s_N; 0)^{\frac{1}{2}}\|.
\]

Hence, cancelling \( \|A^{-\frac{1}{2}} \nabla \cdot s_N\|_N \) and squaring it gives (4.13). Now combining all estimates of (4.12)–(4.14) completes the proof of lower bound in (4.8).

Due to Theorem 4.4, the existence and uniqueness of the problem (4.1) are guaranteed. Note that we allow even discontinuities of \( A \) for the coercivity (4.8). This is one of merits both in adjoint pseudospectral least-squares approaches and in adjoint finite element least-squares approaches. The only assumption on \( A \) is uniformly ellipticity.

Now it is time to get involved in the question of the convergence. Because of (3.16), note that the unique solution of the given first-order system of differential equation (3.2) is given by (see (3.14))

\[
(u, p, q)^t = L^*_{\psi}(w, r, s).
\]

Hence we may consider the approximate solution \( (w_N, r_N, s_N) \) of (4.1) as the approximate solution

\[
(\tilde{u}_N, \tilde{p}_N, \tilde{q}_N)^t = L^*_{\psi}(w_N, r_N, s_N)
\]
corresponding to (3.14). We note that \( (\tilde{u}_N, \tilde{p}_N, \tilde{q}_N) \) may not be of polynomial types because the matrix \( A \) may not be a constant matrix.

First, we will provide the convergence analysis for the approximate solution \( (w_N, r_N, s_N) \) to \( (w, r, s) \). Then we will be back to the convergence of \( (\tilde{u}_N, \tilde{p}_N, \tilde{q}_N) \) to \( (u, p, q) \) later. For the spectral convergence of the approximate solution to the exact solution, we need to assume a sufficient regularity of the exact solution. For this, we suppose that

(A) \( A_{ij}^{(k)} \) and \( A^{-1k}_{ij} \) exist and bounded for \( 0 \leq k \leq d, 1 \leq i, j \leq 2 \) and \( d \geq 2 \).

**Proposition 4.5.** Let \( (w, r, s) \in H^{d+1}(\Omega)^2 \times H^{d+1}(\Omega) \times H^{d+1}(\Omega) \) be the solution of (3.14) for some \( d \geq 2 \). For \n \mathcal{V}_N := (\phi_N, \chi_N, \psi_N) \) and \( \mathcal{W}_N := (\rho_N, \sigma_N, \tau_N) \) in \( \mathcal{T}_N := \mathcal{W}_N \times \mathcal{R}_N \times \mathcal{S}_N \), we have

\[
\left| a_N^{(k)}(\phi_N, \chi_N, \psi_N; (\rho_N, \sigma_N, \tau_N)) - a_N^{(k)}(\phi_N, \chi_N, \psi_N; (\rho_N, \sigma_N, \tau_N)) \right| \\
\leq C_A N^{-d} \left[ \|\phi_N - w\|_{d+1} + \|\chi_N - r\|_{d+1} + \|\psi_N - s\|_{d+1} + \|w\|_{d+1} + \|r\|_{d+1} + \|s\|_{d+1} \right]
\]

where \( C_A \) is a constant dependent on \( A \) and \( A^{-1} \) such that

\[
C_A = C \max \left\{ 1, \max_{0 \leq k \leq d} \|A_{ij}\|, \max_{0 \leq k \leq d} |A^{-1}_{ij}| \right\}.
\]

**Proof.** For convenience, let

\[
f_N := \phi_N - \nabla \cdot \psi_N, \quad g_N := \nabla \chi_N, \\
\hat{f}_N := \rho_N - \nabla \cdot \tau_N, \quad \hat{g}_N := \nabla \sigma_N.
\]

Note that \( \hat{f}_N \) and \( \hat{g}_N \) are polynomials in \( Q_N \) because \( \rho_N, \sigma_N \) and \( \tau_N \) are polynomials in \( Q_N \). Then, using symmetric property of \( A \) and (2.6), one may have for \( d \geq 2 \)

\[
\left| a_N^{(k)}(\phi_N, \chi_N, \psi_N; (\rho_N, \sigma_N, \tau_N)) - a_N^{(k)}(\phi_N, \chi_N, \psi_N; (\rho_N, \sigma_N, \tau_N)) \right| \\
\leq N^{-d} \left( \|A^{-1} f_N - g_N\|_d \|\hat{f}_N\| + \|f_N - A g_N\|_d \|\hat{g}_N\| + \|\nabla \cdot \phi_N\|_d \|\nabla \cdot \rho_N\| + \|\nabla \times \phi_N\|_d \|\nabla \times \rho_N\| \right) \\
\leq C N^{-d} \left( \|A^{-1} f_N\|_d + \|f_N\|_d + \|g_N\|_d + \|A g_N\|_d \right) \|\rho_N, \sigma_N, \tau_N\|_2 \\
+ C N^{-d} \left( \|\nabla \cdot \phi_N\|_d + \|\nabla \times \phi_N\|_d \right) \|\rho_N, \sigma_N, \tau_N\|_2
\]

where \( C \) is a positive constant. Now let us estimate the last right term of (4.19) in terms of the exact solution \( (w, r, s) \in H^3(\Omega)^3 \) of (3.14) for some \( s \geq 3 \). For this, let
Then, with notations $f_N, g_N, \tilde{f}$ and $\tilde{g}$ defined above, we have
\[
\|A^{-1}f_N\|_d + \|f_N\|_d + \|g_N\|_d + \|A^T f_N\|_d \\
\leq \|A^{-1}(f_N - \tilde{f}_N)\|_d + \|f_N - \tilde{f}_N\|_d + \|g_N - \tilde{g}_N\|_d + \|A(g_N - \tilde{g}_N)\|_d \\
+ \|A^{-1}\tilde{f}_N\|_d + \|\tilde{f}_N\|_d + \|\tilde{g}_N\|_d \\
\leq C_A(\|f_N - \tilde{f}_N\|_d + \|g_N - \tilde{g}_N\|_d + \|\tilde{f}_N\|_d + \|\tilde{g}_N\|_d) \\
\leq C_A(\|\phi_N - w\|_d + \|\chi_N - r\|_d + \|\psi_N - s\|_d + \|w\|_d + \|r\|_d + \|s\|_d),
\]
(4.20)
where $C_A = C \max\{1, \max_{0 \leq k \leq d} ||A_{ij}||_k, \max_{0 \leq k \leq d} ||A^{-1}_{ij}||_k\}$ with an absolute positive constant $C$. In a similar way, we have
\[
\|\nabla \phi_N\|_d + \|\nabla \times \phi_N\|_d \leq C(\|\phi_N - w\|_d + \|\chi_N - r\|_d + \|\psi_N - s\|_d),
\]
where $C$ is an absolute positive constant. Now, combining (4.19), (4.20) and (4.21), it follows that
\[
|a^+_e(\phi_N, \chi_N, \psi_N); (\rho_N, \sigma_N, \tau_N)) - a^+_e(N((\phi_N, \chi_N, \psi_N); (\rho_N, \sigma_N, \tau_N)))| \\
\leq C_A N^{-d}(\|\phi_N - w\|_d + \|\chi_N - r\|_d + \|\psi_N - s\|_d) \|\phi_N, \sigma_N, \tau_N\| \mathcal{L}_e^2 \\
+ C_A N^{-d}(\|w\|_d + \|r\|_d + \|s\|_d) \|\phi_N, \sigma_N, \tau_N\| \mathcal{L}_e^2.
\]
These arguments complete the proof. □

We note that in the estimates of (4.19), the error estimates (2.6) between $L^2$ inner product $(u, v_N)$ and discrete inner product $(u, v_N)_N$ is used, which works for $u \in H^d(\Omega)$ where $d \geq 2$ and a polynomial $v_N$ of order $N$.

**Remark 2.** Note that if $A$ is a constant matrix, then the estimate (4.17) will be vanished if $(\rho_N, \sigma_N, \tau_N)$ is chosen in $T_{N-1}$ because of the exactness of numerical quadrature. This observation shows us that the spectral convergence of the least-squares adjoint approach is not deteriorated. That is to say, it means that the least-squares adjoint approach has the same spectral convergence as occurred in usual least-squares approach (see [13]) for the problem with constant coefficients.

**Theorem 4.6.** Let $(w, r, s) \in H^{d+1}(\Omega) \times H^{d+1}(\Omega) \times H^{d+1}(\Omega)$ be the solution of (3.14) for some $d \geq 2$ and $f \in H^l(\Omega)$ for some integer $l \geq 2$. Let $(w_N, r_N, s_N) \in T_{N} := W_N \times R_N \times S_N$ be the discrete solution of the problem (4.1). Then there exists a positive constant $C$ such that
\[
C(\|w - w_N\|_1 + \|r - r_N\|_1 + \|s - s_N\|_1) \\
\leq \inf_{(\phi_N, \chi_N, \psi_N) \in T_N} \left[\|w - \phi_N\|_1 + \|r - \chi_N\|_1 + \|s - \psi_N\|_1\right] \\
+ N^{-d}(\|w - \phi_N\|_d + \|r - \chi_N\|_d + \|s - \psi_N\|_d) \\
+ N^{-d}(\|w\|_d + \|r\|_d + \|s\|_d) + N^{-l} \|f\|_l.
\]
(4.23)

**Proof.** Using the first Strang lemma which uses the coercivity of $a^+_e(\cdot, \cdot, \cdot)$ and the continuity of $a^+_e(\cdot, \cdot, \cdot)$, one may easily verify from [14] (or see p. 88 in [11]) that
\[
C(\|w - w_N\|_1 + \|r - r_N\|_1 + \|s - s_N\|_1) \\
\leq \inf_{(\phi_N, \chi_N, \psi_N) \in W_N \times R_N \times S_N} \left[\|w - \phi_N\|_1 + \|r - \chi_N\|_1 + \|s - \psi_N\|_1\right] \\
+ \sup_{(\rho_N, \sigma_N, \tau_N) \in W_N \times R_N \times S_N} \left[\|a^+_e((\phi, \chi, \psi); (\rho, \sigma, \tau)) - a^+_e((\phi_N, \chi_N, \psi_N); (\rho_N, \sigma_N, \tau_N))\|_1\right] \\
+ \sup_{(\rho, \sigma, \tau) \in W_N \times R_N \times S_N} \left[\|0 - f, 0\|_1 \|\rho, \sigma, \tau\|_1\right].
\]
Using the inequality (2.6), the third term of left side of (4.24) can be estimated as

$$\left| \left( (0, -f, 0), (\rho, \sigma, \tau) \right) - \left( (0, -f, 0), (\rho, \sigma, \tau) \right) \right|_N = \left| (f, \sigma) - (f, \sigma) \right|_N \leq CN^{-\ell} f ||\sigma||, \quad \ell \geq 2. \quad (4.28)$$

The estimate for the second term of right side of (4.24) is estimated in Proposition 4.5. Hence combining (4.28) and (4.17), we deduce the conclusion (4.23). □

Note that using the approximation results (2.7) yields the convergence estimates (4.30). However one may use the interpolation error estimates found in [7] as

$$\|u - I_Nu\|_k \leq CN^{2k - d + \frac{3}{2}} \|u\|_d, \quad (4.29)$$

where $d > 1$ and $0 \leq k \leq d$. But the optimality of this approximation is not known (see Theorem 3.2 in [7]) and leads to a worse convergence estimate. For $k = 0, 1$, the optimal error estimate is given in (2.5) for $d \geq 2$.

**Corollary 4.7.** Assume that (A) holds. Let $(w, r, s) \in H^m(\Omega)^2 \times H^m(\Omega) \times H^m(\Omega)$ be the solution of (3.14) where $m > d + \frac{3}{2}$ and $d \geq 2$. Let $f \in H^\ell(\Omega)$ for some integer $\ell \geq 2$. Let $(w_N, r_N, s_N) \in T_N := W_N \times R_N \times S_N$ be the discrete solution of the problem (4.1). It follows that with a positive constant $C$

$$C(\|w - w_N\|_1 + \|r - r_N\|_1 + \|s - s_N\|_1) \leq N^{d + \frac{3}{2} - m}(\|w\|_m + \|r\|_m + \|s\|_m) + N^{-\ell} f \|l\|. \quad (4.30)$$

**Proof.** Using the approximation result (2.7) for (4.23), one may easily show the conclusion (4.30). □

**Remark 3.** We note that with a more regularity for the solution $(w, r, s) \in H^m(\Omega)^2 \times H^m(\Omega) \times H^m(\Omega)$ with $m > 2d + \frac{3}{2}$ and $d \geq 2$ in the above corollary, one can get $N^{-d}$ instead of $N^{d - m + \frac{3}{2}}$.

Now, due to Corollary 4.7 we can provide the convergence result. From the theory of finite element approximations for (4.15) and (4.16) using high-order elements, one may have the first inequality below and one may have the second inequality because of continuity of $L_e^*$ due to Theorem 3.1 below:

$$\|(u, p, q) - (\tilde{u}_N, \tilde{p}_N, \tilde{q}_N)\|_{L_e^*} \leq \inf_{(w_N, r_N, s_N) \in W_N \times R_N \times S_N} \| L_e^* (u, p, q) - (w_N, r_N, s_N) \|_{L_e^*}$$

$$\leq C \inf_{(w_N, r_N, s_N) \in W_N \times R_N \times S_N} \| (w, r, s) - (w_N, r_N, s_N) \|_{L_e^*}$$

$$\leq C N^{d + \frac{3}{2} - m}(\|w\|_m + \|r\|_m + \|s\|_m) + N^{-\ell} f \|l\|. \quad (4.31)$$

**5. Numerical tests**

In this section, we present some numerical experiments for the first-order system corresponding to (1.1). A mixed boundary condition will be considered also. Note that, because of the solution $(w, r, s)^T \in D^\theta$ of (3.14) and (4.15), $p = \nabla \cdot w - cr$ is the solution of (1.1) and the slack variable $q = -\nabla \times w$ becomes zero due to Theorem 3.1. Hence we will provide the errors $e_u = u - \tilde{u}_N$ and $e_p = p - \tilde{p}_N$ where $(\tilde{u}_N, \tilde{p}_N)$ is defined in (4.16) as

$$\tilde{u}_N = A^{-\frac{1}{2}}w_N - A^{\frac{1}{2}}r_N - A^{\frac{1}{2}}b r_N - A^{\frac{1}{2}}s N \quad \text{and} \quad \tilde{p}_N = \nabla \cdot w_N - cr_N.$$

Denote by $|e_v|_{1,N} := \|\nabla e_v\|_N$ for $v = p$ or $u$. Using the spectral convergence property that

$$E_N \approx CN^{-\rho} \quad \text{and} \quad \frac{E_N}{E_{2N}} \approx \frac{CN^{-\rho}}{C(N)^{-\rho}} = 2^\rho,$$

we define the rate of convergence $\rho$ as

$$\rho = \log_2 \frac{E_N}{E_{2N}}$$

where $E_N$ denotes $\|e_p\|_N$, $|e_p|_{1,N}$, $\|e_u\|_N$ or $|e_u|_{1,N}$. In our experiments, we have checked the rate of convergence with $N = 12$. 

\[ \text{S.D. Kim, B.-C. Shin} / \text{Applied Numerical Mathematics 59 (2009) 334–348} \]
where the diffusion coefficient matrix is given by

\[ A = \begin{pmatrix} \cos xy + e^{xy} & \cos xy - e^{xy} \\ \cos xy - e^{xy} & \cos xy + e^{xy} \end{pmatrix}. \]

Here, the right-hand side \( f \) is given by the differential equation with the exact solution

\[ p(x, y) = e^{xy} \sin 2\pi x \sin 2\pi y. \]

Since all elements of the diffusion matrix \( A \) and \( A^{-1} \) are of class \( C^\infty \) in \( \Omega \), the assumption (A) is satisfied. Hence the convergence is guaranteed by Corollary 4.7. Actually, the numerical results in Fig. 1 and Table 1 support the spectral convergence for both variables \( p \) and \( u \) in the discrete \( L^2 \)- and \( H^1 \)-semi norms, that is, the errors of all variables in each norm decay exponentially. Even if the nonnegativity on \( c \) is assumed in the section of introduction, our approach can be performed well for \( c = -10 \).

**Example 2.** In this example we will present the numerical experiments with a diffusion coefficient matrix \( A_{\alpha} = \lambda_{\alpha}(x, y) I \) with \( \lambda_{\alpha} \in H^\alpha(\Omega) \) for \( \alpha = 2, 4, 6 \). This example shows that the rate of convergence is dependent on the regularity of the coefficient matrix \( A_{\alpha} \). We now consider the following diffusion problem:

\[ \begin{cases} -\nabla \cdot A_{\alpha} \nabla p = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial \Omega. \end{cases} \]

Here, we take the diffusion coefficient matrix \( A_{\alpha} = \lambda_{\alpha}(x, y) I \) with

\[ \lambda_{\alpha}(x, y) = \begin{cases} e^{xy+1} & \text{for } -1 \leq x \leq 0, \\ e^{xy+1} + x^\alpha & \text{for } 0 \leq x \leq 1. \end{cases} \]

Note that \( \lambda_{\alpha} \in H^\alpha(\Omega) \) for \( \alpha \geq 1 \). Also, the right-hand side \( f \) is given by the differential equation with the exact solution
These numerical tests in Table 2 and Fig. 2 reveal that the rate of decay of the spectral error is graded by increasing order or regularity $\alpha$ of the coefficient matrix $A_{\alpha}$, as predicted by the convergence theory given in (4.31).

Table 2
Discretization errors and rates of convergence for Example 2

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$N$</th>
<th>$|e_p|_N$</th>
<th>$|\epsilon_p|_{1,N}$</th>
<th>$|e_u|_N$</th>
<th>$|\epsilon_u|_{1,N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3.1616e+00</td>
<td>9.7835e+00</td>
<td>4.6986e+01</td>
<td>2.1751e+02</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>8.0547e+02</td>
<td>9.9777e+01</td>
<td>4.1593e+00</td>
<td>6.5632e+01</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>2.0792e+03</td>
<td>3.1210e+02</td>
<td>5.3868e+02</td>
<td>1.3166e+00</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>6.1505e+04</td>
<td>6.3534e+03</td>
<td>1.4774e+02</td>
<td>3.3741e-01</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>3.5054e+04</td>
<td>3.5791e+03</td>
<td>8.1372e-03</td>
<td>1.9678e-01</td>
</tr>
<tr>
<td></td>
<td>24</td>
<td>2.3010e+04</td>
<td>2.3290e+03</td>
<td>5.2609e-03</td>
<td>1.3655e-01</td>
</tr>
<tr>
<td>Rate of convergence $\rho$</td>
<td>3.16</td>
<td>3.74</td>
<td>3.36</td>
<td>3.27</td>
<td></td>
</tr>
</tbody>
</table>

| 4        | 4   | 3.1510e+00  | 9.6433e+00      | 4.6069e+01  | 2.1327e+02      |
|          | 8   | 8.1151e+02  | 1.0087e+00      | 4.1495e+00  | 6.4311e+01      |
|          | 12  | 1.3698e+03  | 2.7215e+02      | 3.6695e-02  | 9.1216e-01      |
|          | 16  | 2.1810e+05  | 4.2281e+04      | 9.9271e-04  | 3.9854e-02      |
|          | 20  | 4.3428e-06  | 8.9895e-05      | 1.9585e-04  | 8.7802e-03      |
|          | 24  | 1.4649e-06  | 3.1755e-05      | 6.5891e-05  | 3.3033e-03      |
| Rate of convergence $\rho$ | 9.87 | 9.74 | 9.12 | 8.11 |

| 6        | 4   | 3.1535e+00  | 9.6116e+00      | 4.5741e+01  | 2.1170e+02      |
|          | 8   | 8.0712e+02  | 1.0086e+00      | 4.1591e-01  | 6.4295e+01      |
|          | 12  | 1.3794e-03  | 2.6656e-02      | 3.5188e-02  | 9.9941e-01      |
|          | 16  | 8.8272e-06  | 1.9373e-04      | 4.5644e-04  | 2.1928e-02      |
|          | 20  | 4.8459e-07  | 1.4673e-05      | 2.9337e-05  | 1.6876e-03      |
|          | 24  | 7.6182e-08  | 2.8150e-06      | 4.8830e-06  | 3.1510e-04      |
| Rate of convergence $\rho$ | 14.14 | 13.21 | 12.81 | 11.63 |

$p(x, y) = e^{\alpha y} \sin 2\pi x \sin 2\pi y$.

These numerical tests in Table 2 and Fig. 2 reveal that the rate of decay of the spectral error is graded by increasing order or regularity $\alpha$ of the coefficient matrix $A_{\alpha}$, as predicted by the convergence theory given in (4.31).

**Example 3.** In this example we will provide our numerical experiments (see Table 3 and Fig. 3) for an elliptic problem with mixed boundary condition. Consider the following elliptic problem

\[
\begin{aligned}
\begin{cases}
-\nabla \cdot A\nabla p + b \cdot \nabla p + \alpha p &= f &\text{in } \Omega, \\
p &= 0 &\text{on } \Gamma_D, \\
n \cdot \nabla p &= 0 &\text{on } \Gamma_N,
\end{cases}
\end{aligned}
\]
of approach with pseudospectral methods to solve elliptic boundary value problems with variable diffusion coefficient.

6. Conclusion

The questions on stability, uniqueness and convergence are studied in this paper for the adjoint least-squares approach with pseudospectral methods to solve elliptic boundary value problems with variable diffusion coefficient.

Table 3

<table>
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<tr>
<th>b</th>
<th>c</th>
<th>N</th>
<th>|e_p|_N</th>
<th>|e_p|_{1,N}</th>
<th>|e_u|_N</th>
<th>|e_u|_{1,N}</th>
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<td>11.38</td>
<td>10.91</td>
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<td>10.65</td>
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</table>

where \( \Gamma_D := ((-1) \times [-1, 1]) \cup ((-1, 1) \times \{-1\}) \) and \( \Gamma_N = \partial \Omega \setminus \Gamma_D \). We had tested our approaches with \( A = A_6 \) given in Example 2 and the smooth exact solution

\[
p(x, y) = \sin \left( \frac{7\pi (x + 1)}{4} \right) \sin \left( \frac{7\pi (y + 1)}{4} \right).
\]

For the mixed boundary value problem, we have very similar results as those of Example 1.
functions by combining the ideas of [5] and [13]. In particular, in the proofs of the existence and uniqueness of the approximated polynomial solution \((w_N, r_N, s_N)\) (accordingly \((\tilde{u}_N, \tilde{p}_N, \tilde{q}_N)\) in (4.16)), it is not necessary to compute derivatives of the diffusion coefficient as seen in Theorem 4.4. On the other hand, to get a spectral convergence for the proposed scheme, the sufficient regularities of the solutions are required (see Theorem 4.6), which may be a general restriction for spectral methods (see [1] for example). It is still remained to analyze the convergence under mild regularities of the solutions. As seen in Examples 1 and 3 in Section 5, the convergence phenomena are as good as the usual least-squares pseudospectral methods (see [13]).

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References